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A complete set of local invariants for a family of multipartite mixed states

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Abstract
We study the equivalence of quantum states under local unitary transformations by using the singular value decomposition. A complete set of invariants under local unitary transformations is presented for several classes of tripartite mixed states in $\mathbb{C}^K \otimes \mathbb{C}^M \otimes \mathbb{C}^N$ composite systems. Two density matrices in the same class are equivalent under local unitary transformations if and only if all these invariants have equal values for these density matrices.

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1. Introduction

Quantum entanglement is playing a very important role in quantum information processing. Quantum entangled states are the key resource in quantum information processing [1] such as teleportation, super-dense coding, key distribution, error correction and quantum repeaters. Therefore, it is of great importance to classify and characterize the quantum states.

The nature of the entanglement among the parts of a composite system does not depend on the labeling of the basis states of the individual subsystems. It is therefore invariant under unitary transformations of the individual state spaces. Such transformations are referred to as local unitary transformations. The polynomial invariants of local unitary transformations have been discussed in [2–4]. General methods, which allow us in principle to compute all such invariants, but are in fact not really operational, were introduced in [5–8]. More explicit complete and partial solutions have been found for some special cases: two qubit [9] and three qubit [10, 11] systems, three qutrits [12], generic mixed states [13] and special families of tripartite pure qudits [14–16].

The problem of classifying states under local unitary transformations can be solved completely for bipartite pure states. As the set of Schmidt coefficients forms a complete set of invariants under local unitary transformations, two bipartite pure states are equivalent under local unitary transformations if and only if they have the same Schmidt coefficients. For a multiple composite system, there does not exist Schmidt decomposition in general. There are different generalizations for Schmidt decomposition in multipartite quantum pure states [17–21], but the results are not sufficient to solve the local equivalence problem. For multipartite mixed states, much less is known about the equivalence under local unitary transformations.

Another classification of quantum states is the one under stochastic local operations and classical communications (SLOCC). Invariants under SLOCC have also been extensively studied [22–24]. Recently, Lamata *et al* [25] used the method of singular value decomposition and presented an inductive classification of multipartite qubit systems under SLOCC.

In this paper, we study the equivalence of multipartite mixed states under local unitary transformations by using the singular value decomposition. Let $\mathcal{H}_1$ (resp. $\mathcal{H}_2$) be M (resp. N) dimensional complex Hilbert spaces ($M \leqslant N$). A mixed state ρ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ with rank $r(\rho) = n \leqslant M^2$ can be decomposed according to its eigenvalues λ_i and eigenvectors $|\nu_i\rangle, i = 1, \ldots, n$:

$$\rho = \sum_{i=1}^{n} \lambda_i |\nu_i\rangle\langle\nu_i|.$$

In [26], a class of bipartite mixed states Γ_0 has been defined; Γ_0 contains all the states ρ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfying

$$[\rho_i, \rho_j] = 0, \qquad [\theta_i, \theta_j] = 0, \qquad i, j = 1, 2, \ldots, n, \tag{1}$$

where ρ_i are full rank matrices,

$$\rho_i = \mathrm{Tr}_2|\nu_i\rangle\langle\nu_i|, \qquad \theta_i = (\mathrm{Tr}_1|\nu_i\rangle\langle\nu_i|)^*, \qquad i = 1, \ldots, n,$$

and Tr_1 (resp. Tr_2) denotes the partial trace over $\mathcal{H}_1$ (resp. $\mathcal{H}_2$). We denote by †, * and ^{t} the adjoint, complex conjugation and transposition, respectively.

It has been shown that two mixed states in Γ_0 are equivalent under local unitary transformations if and only if the following invariants ((a) or (b)) have the same values for both mixed states [26]:

$$(a) \quad \mathrm{Tr}\big(\rho_i^\alpha\big), \quad \mathrm{Tr}(\rho^\gamma), \quad \alpha = 1, 2, \ldots, M, \quad \gamma = 1, 2, \ldots, MN.$$

$$(b) \quad \mathrm{Tr}(\theta_i^\beta), \quad \mathrm{Tr}(\rho^\gamma), \quad \beta = 1, 2, \ldots, N, \quad \gamma = 1, 2, \ldots, MN.$$

The set of such states in Γ_0 is not trivial. In fact, Γ_0 is a subset of the Schmidt-correlated (SC) states [27]. The SC states are defined as mixtures of pure states, sharing the same Schmidt bases. It first appeared in [28], named as maximally correlated state. For SC states, for any classical measurement, two observers Alice and Bob will always obtain the same result. Two SC states can always be optimally discriminated locally. It is interesting that maximally entangled states (Bell state) can always be expressed in Schmidt-correlated form. SC states naturally appear in a bipartite system dynamics with additive integrals of motion [29]. Hence, these states form an important class of mixed states from a quantum dynamical perspective. From the definition of SC state, we know that the states in Γ_0 are all SC states. Therefore, we can judge whether a state in Γ_0 is separable or not by calculating the negativity of this state [30].

Here we give a simple way to construct some families of states in Γ_0. For $M = N = 4$, one can set $|\psi_1\rangle = (|00\rangle + |12\rangle + |21\rangle + |33\rangle)/2$ and $|\psi_2\rangle = (|01\rangle + |10\rangle + |23\rangle + |32\rangle)/2$, where $|ij\rangle, i = 0, 1, \ldots, M-1, j = 0, 1, \ldots, N-1$, are the basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Then $\rho = \alpha|\psi_1\rangle\langle\psi_1| + (1-\alpha)|\psi_2\rangle\langle\psi_2|$ is a rank-two state belonging to Γ_0 for $0 < \alpha < 1$. For general even $M = N = d+1$, a state $\rho = \alpha|\psi_1\rangle\langle\psi_1| + (1-\alpha)|\psi_2\rangle\langle\psi_2|$ is in Γ_0, where $|\psi_1\rangle = (|00\rangle + |12\rangle + |21\rangle + |34\rangle + |43\rangle + \cdots + |dd\rangle)/\sqrt{M}$ and $|\psi_2\rangle = (|01\rangle + |10\rangle + |23\rangle + |32\rangle + \cdots + |d-1, d\rangle + |d, d-1\rangle)/\sqrt{M}$.

For $M = N = 5$, one can set $|\phi_1\rangle = (|00\rangle + |12\rangle + |21\rangle + |34\rangle + |43\rangle)/\sqrt{5}$ and $|\phi_2\rangle = (|01\rangle + |10\rangle + |23\rangle + |32\rangle + |44\rangle)/\sqrt{5}$. Then $\rho = \alpha|\phi_1\rangle\langle\phi_1| + (1-\alpha)|\phi_2\rangle\langle\phi_2|$ is a rank-two state in Γ_0. For general odd $M = N$, $|\phi_1\rangle$ and $|\phi_2\rangle$ can be similarly constructed.

We can also construct higher rank states in Γ_0. For example, for $M = N = 4$, by adding $|\psi_3\rangle = (|11\rangle + |02\rangle + |20\rangle + |33\rangle)/2$, we have that $\rho = \alpha|\psi_1\rangle\langle\psi_1| + \beta|\psi_2\rangle\langle\psi_2| + (1-\alpha-\beta)|\psi_3\rangle\langle\psi_3|$ is a state in Γ_0. For odd $M = N = 5$, we have $|\phi_3\rangle = (|04\rangle + |13\rangle + |22\rangle + |31\rangle + |40\rangle)/\sqrt{5}$ and $\rho = \alpha|\phi_1\rangle\langle\phi_1| + \beta|\phi_2\rangle\langle\phi_2| + (1-\alpha-\beta)|\phi_3\rangle\langle\phi_3| \in \Gamma_0$.

The states constructed above are all distillable. The rank of reduced density matrices, which are in fact identity matrices, are greater than the rank of ρ itself. They are all NPPT (non-positive partial transpose) entangled states.

2. Tripartite quantum pure states

We first discuss the locally invariant properties of arbitrary dimensional tripartite pure quantum states. Let $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ be K-, M- and N-dimensional complex Hilbert spaces with the orthonormal bases $\{|e_i\rangle\}_{i=1}^{K}$, $\{|f_i\rangle\}_{i=1}^{M}$ and $\{|h_i\rangle\}_{i=1}^{N}$, respectively.

$|\Psi\rangle$ can be regarded as a bipartite state by taking $\mathcal{H}_1$ (resp. $\mathcal{H}_2, \mathcal{H}_3$) and $\mathcal{H}_2 \otimes \mathcal{H}_3$ (resp. $\mathcal{H}_1 \otimes \mathcal{H}_3, \mathcal{H}_1 \otimes \mathcal{H}_2$) as the two subsystems. We denote these three bipartite decompositions as 1–23 (resp. 2–13, 3–12). Let a_{ijk} be the coefficients of $|\Psi\rangle$ in orthonormal bases $|e_i\rangle \otimes |f_j\rangle \otimes |h_k\rangle$. Let A_1 (resp. A_2, A_3) denote the matrix with respect to the bipartite state in 1–23 (resp. 2–13, 3–12) decomposition, i.e. taking the subindices i (resp. j, k) and jk (resp. ik, ij) of a_{ijk} as the row and column indices of A_1 (resp. A_2, A_3).

Taking partial trace of $|\Psi\rangle\langle\Psi|$ over the respective subsystems, we have $\tau_1 = \mathrm{Tr}_1|\Psi\rangle\langle\Psi| = A_1^t A_1^*$, $\tau_2 = \mathrm{Tr}_2|\Psi\rangle\langle\Psi| = A_2^t A_2^*$ and $\tau_3 = \mathrm{Tr}_3|\Psi\rangle\langle\Psi| = A_3^t A_3^*$. The reduced matrices τ_1, τ_2 and τ_3 can be decomposed according to their eigenvalues and eigenvectors, e.g.,

$$\tau_1 = \sum_{i=1}^{n_1} \lambda_i^1 \big|\nu_i^1\big\rangle\big\langle\nu_i^1\big|,$$

where λ_i^1, resp. $\big|\nu_i^1\big\rangle$, $i = 1, \ldots, n_1$, are the nonzero eigenvalues, resp. eigenvectors, of the density matrix τ_1.

Let A_i^1 denote the matrix with entries given by the coefficients of $\big|\nu_i^1\big\rangle$ in the bases $|f_k\rangle \otimes |h_l\rangle$. We have

$$\rho_i^1 = \mathrm{Tr}_3\big|\nu_i^1\big\rangle\big\langle\nu_i^1\big| = A_i^1 A_i^{1\dagger}, \qquad \theta_i^1 = \big(\mathrm{Tr}_2\big|\nu_i^1\big\rangle\big\langle\nu_i^1\big|\big)^* = A_i^{1\dagger} A_i^1, \qquad i = 1, \ldots, n_1.$$

Set

$$\begin{aligned} I_\alpha^1(|\Psi\rangle) &= \mathrm{Tr}\big(\rho_i^1\big)^\alpha, &\qquad \alpha &= 1, 2, \ldots, M,\\ J_\beta^1(|\Psi\rangle) &= \mathrm{Tr}\big(\theta_i^1\big)^\beta, &\qquad \beta &= 1, 2, \ldots, N,\\ K_\gamma^1(|\Psi\rangle) &= \mathrm{Tr}\big(\tau_1^\gamma\big), &\qquad \gamma &= 1, 2, \ldots, MN. \end{aligned}$$

It is easy to prove that $I_\alpha^1(|\Psi\rangle)$, $J_\beta^1(|\Psi\rangle)$ and $K_\gamma^1(|\Psi\rangle)$ are all invariants under local unitary transformations.

Let Γ_1 denote a class of tripartite pure states $|\Psi\rangle$ satisfying

$$\big[\rho_i^1, \rho_j^1\big] = 0, \qquad \big[\theta_i^1, \theta_j^1\big] = 0 \tag{2}$$

with ρ_i^1 being full rank matrices, $i, j = 1, 2, \ldots, n_1$.

Theorem 1. *Two pure states in* Γ_1 *are equivalent under local unitary transformations if and only if the following invariants ((c) or (d)) have the same values for both states:*

$$(c)\quad I_\alpha^1(|\Psi\rangle),\qquad K_\gamma^1(|\Psi\rangle),\qquad \alpha = 1, 2, \ldots, M,\qquad \gamma = 1, 2, \ldots, MN.$$

$$(d)\quad J_\beta^1(|\Psi\rangle),\qquad K_\gamma^1(|\Psi\rangle),\qquad \beta = 1, 2, \ldots, N,\qquad \gamma = 1, 2, \ldots, MN.$$

We only need to prove the sufficient part. Assume $|\Psi\rangle, |\Psi'\rangle \in \Gamma_1$. $K_\gamma^1(|\Psi\rangle) = K_\gamma^1(|\Psi'\rangle)$ *imply that* A_1 *and* A'_1 *have the same singular values, therefore there exist unitary matrices* U_1 *and* U_{23} *such that* $|\Psi'\rangle = U_1 \otimes U_{23}|\Psi\rangle$. *If* $I_\alpha^1(|\Psi\rangle) = I_\alpha^1(|\Psi'\rangle)$ *or* $J_\beta^1(|\Psi\rangle) = J_\beta^1(|\Psi'\rangle)$ *holds, then* τ_1 *and* τ'_1 *are equivalent under local unitary transformations by the sufficient condition of equivalence for bipartite states under local unitary transformations. While in [15] it has been proven that if* $|\Psi'\rangle = U_1 \otimes U_{23}|\Psi\rangle$, *with* $U_1 \in U(\mathcal{H}_1)$, $U_{23} \in U(\mathcal{H}_2 \otimes \mathcal{H}_3)$ *and* $\mathrm{Tr}_1(|\Psi'\rangle\langle\Psi'|) = U_2 \otimes U_3 \mathrm{Tr}_1(|\Psi\rangle\langle\Psi|)U_2^\dagger \otimes U_3^\dagger$, *where* $U_2 \in U(\mathcal{H}_2)$ *and* $U_3 \in U(\mathcal{H}_3)$, *then there exist matrices* $V_1 \in U(\mathcal{H}_1)$, $V_2 \in U(\mathcal{H}_2)$, $V_3 \in U(\mathcal{H}_3)$ *such that* $|\Psi'\rangle = V_1 \otimes V_2 \otimes V_3|\Psi\rangle$, *i.e.,* $|\Psi\rangle$ *and* $|\Psi'\rangle$ *are equivalent under local unitary transformations.*

Let us consider for example two states $|\Psi\rangle = \sqrt{\frac{p}{3}}(|000\rangle + |012\rangle + |021\rangle) + \sqrt{\frac{1-p}{3}}(|101\rangle + |110\rangle + |122\rangle)$ and $|\Psi'\rangle = \sqrt{\frac{p}{3}}(|000\rangle + |011\rangle + |022\rangle) + \sqrt{\frac{1-p}{3}}(|101\rangle + |112\rangle + |120\rangle)$ in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, for the case $K = 2, M = N = 3$. It is direct to verify that they are all states in Γ_1 with $\rho_i^1 = \theta_i^1 = \frac{1}{3}I, i = 1, 2$. As τ_1 and τ'_1 have the same eigenvalues, relation $K_\gamma^1(|\Psi\rangle) = K_\gamma^1(|\Psi'\rangle)$ holds, from which and from the following equations

$$\mathrm{Tr}\big(\rho_i^1\big) = \mathrm{Tr}\big(\rho_i'^1\big) = 1, \qquad \mathrm{Tr}\big(\rho_i^1\big)^2 = \mathrm{Tr}\big(\rho_i'^1\big)^2 = \tfrac{1}{3},$$

by theorem 1 we have that $|\Psi\rangle$ and $|\Psi'\rangle$ are equivalent under local unitary transformations. The same results can also be obtained from $K_\gamma^1(|\Psi\rangle) = K_\gamma^1(|\Psi'\rangle)$ and the following facts:

$$\mathrm{Tr}\big(\theta_i^1\big) = \mathrm{Tr}\big(\theta_i'^1\big) = 1, \qquad \mathrm{Tr}\big(\theta_i^1\big)^2 = \mathrm{Tr}\big(\theta_i'^1\big)^2 = \tfrac{1}{3}.$$

As an alternative example we consider two states $|\Psi\rangle = \sqrt{\frac{\alpha}{3}}(|000\rangle + |012\rangle + |021\rangle) + \sqrt{\frac{\beta}{3}}(|101\rangle + |110\rangle + |122\rangle) + \sqrt{\frac{\gamma}{3}}(|202\rangle + |211\rangle + |220\rangle)$ and $|\Psi'\rangle = \sqrt{\frac{\alpha}{3}}(|000\rangle + |011\rangle + |022\rangle) + \sqrt{\frac{\beta}{3}}(|101\rangle + |112\rangle + |120\rangle) + \sqrt{\frac{\gamma}{3}}(|202\rangle + |210\rangle + |221\rangle)$ in$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, with $K = M = N = 3, \alpha, \beta, \gamma \in R, \alpha + \beta + \gamma = 1$. One can prove that they are all states in Γ_1 with $\rho_i^1 = \theta_i^1 = \frac{1}{3}I, i = 1, 2, 3$, and τ_1, τ'_1 have the same eigenvalues. As

$$\mathrm{Tr}\big(\rho_i^1\big) = \mathrm{Tr}\big(\rho_i'^1\big) = 1, \qquad \mathrm{Tr}\big(\rho_i^1\big)^2 = \mathrm{Tr}\big(\rho_i'^1\big)^2 = \tfrac{1}{3}, \qquad \mathrm{Tr}\big(\rho_i^1\big)^3 = \mathrm{Tr}\big(\rho_i'^1\big)^3 = \tfrac{1}{9},$$

from theorem 1 we have that $|\Psi\rangle$ and $|\Psi'\rangle$ are equivalent under local unitary transformations. Moreover by using the generalized concurrence [31], we have $C_3^3 \neq 0$, hence $|\Psi\rangle$ and $|\Psi'\rangle$ are entangled.

Remark. We can also similarly define the set of states Γ_2. Let τ_2 be a reduced density matrix by tracing $|\Psi\rangle\langle\Psi|$ over the second system. τ_2 can be decomposed according to its eigenvalues and eigenvectors:

$$\tau_2 = \sum_{i=1}^{n_2} \lambda_i^2 \big|\nu_i^2\big\rangle\big\langle\nu_i^2\big|,$$

where λ_i^2, resp. $\big|\nu_i^2\big\rangle, i = 1, \ldots, n_2$, are the nonzero eigenvalues, resp. eigenvectors, of the density matrix τ_2. Define $\big\{\rho_i^2\big\}, \big\{\theta_i^2\big\}$,

$$\rho_i^2 = \mathrm{Tr}_3\big|\nu_i^2\big\rangle\big\langle\nu_i^2\big|, \qquad \theta_i^2 = \big(\mathrm{Tr}_1\big|\nu_i^2\big\rangle\big\langle\nu_i^2\big|\big)^*, \qquad i = 1, \ldots, n_2.$$

We define Γ_2 to be a set of tripartite pure states $|\Psi\rangle$ satisfying

$$\left[\rho_i^2, \rho_j^2\right] = 0, \qquad \left[\theta_i^2, \theta_j^2\right] = 0 \tag{3}$$

with ρ_i^2 being full rank matrices. Then we also have the similar result.

Theorem 2. *Two pure states in Γ_2 are equivalent under local unitary transformations if and only if the following invariants ((e) or (f)) have the same values for both states:*

$$(e) \quad I_\alpha^2(|\Psi\rangle), \quad K_\gamma^2(|\Psi\rangle), \quad \alpha = 1, 2, \ldots, K, \quad \gamma = 1, 2, \ldots, KN,$$
$$(f) \quad J_\beta^2(|\Psi\rangle), \quad K_\gamma^2(|\Psi\rangle), \quad \beta = 1, 2, \ldots, N, \quad \gamma = 1, 2, \ldots, KN,$$

where $I_\alpha^2(|\Psi\rangle) = \mathrm{Tr}\big(\rho_i^2\big)^\alpha$, $J_\beta^2(|\Psi\rangle) = \mathrm{Tr}\big(\theta_i^2\big)^\beta$, $K_\gamma^2(|\Psi\rangle) = \mathrm{Tr}\big(\tau_2^\gamma\big)$.

The set of states Γ_3 can be defined in a similar way and the corresponding theorem (like theorems 1 and 2) can be obtained similarly.

The above results can be generalized to general many partite systems. As each n partite pure states can be treated as a bipartite one, the jth system and rest $n-1$ partite system, by using the results of lemma 2 in [15], one can similarly obtain a complete set of invariants for some classes of multipartite pure states.

3. Tripartite quantum mixed states

We consider now mixed states in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. We assume $K \leqslant M, N$. Let ρ be a density matrix defined on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$ with $r(\rho) = n \leqslant K^3$. ρ can be decomposed according to its eigenvalues and eigenvectors:

$$\rho = \sum_{i=1}^{n} \lambda_i |\nu_i\rangle\langle\nu_i|,$$

where λ_i, resp. $|\nu_i\rangle, i = 1, \ldots, n$, are the nonzero eigenvalues, resp. eigenvectors, of the density matrix ρ. We introduce

$$\rho_i = \mathrm{Tr}_1|\nu_i\rangle\langle\nu_i|, \qquad \theta_i = \mathrm{Tr}_2|\nu_i\rangle\langle\nu_i|, \qquad \gamma_i = \mathrm{Tr}_3|\nu_i\rangle\langle\nu_i|.$$

If we treat $|\nu_i\rangle$ as a bipartite state $|\omega_i\rangle$ in $1-23$ system, let A_{1i} denote the matrix with entries given by the coefficients of $|\omega_i\rangle$ in the bases $|e_k\rangle \otimes |g_l\rangle$, where $|g_l\rangle = |f_t\rangle \otimes |h_s\rangle, l = ts$; $t = 1, \ldots, M, s = 1, \ldots, N$. According to the result of bipartite system, we have

$$\mathrm{Tr}_2|\omega_i\rangle\langle\omega_i| = A_{1i}A_{1i}^\dagger, \qquad (\mathrm{Tr}_1|\omega_i\rangle\langle\omega_i|)^* = A_{1i}^\dagger A_{1i}, \qquad i = 1, \ldots, n.$$

As $\mathrm{Tr}_2|\omega_i\rangle\langle\omega_i| = \mathrm{Tr}_3(\mathrm{Tr}_2|\nu_i\rangle\langle\nu_i|)$ and $\mathrm{Tr}_1|\omega_i\rangle\langle\omega_i| = \mathrm{Tr}_1|\nu_i\rangle\langle\nu_i|$, we have

$$\theta_i^{23} = A_{1i}A_{1i}^\dagger, \qquad \rho_i = \big(A_{1i}^\dagger A_{1i}\big)^*,$$

where $\theta_i^{23} = \mathrm{Tr}_3(\mathrm{Tr}_2|\nu_i\rangle\langle\nu_i|)$.

ρ_i can be again decomposed according to its eigenvalues and eigenvectors:

$$\rho_i = \sum_{j=1}^{m_i} \alpha_j^i \big|\mu_j^i\big\rangle\big\langle\mu_j^i\big|,$$

where α_j^i, resp. $\big|\mu_j^i\big\rangle$, $j = 1, \ldots, m_i$, are the nonzero eigenvalues, resp. eigenvectors, of the reduced density matrix ρ_i. Let B_j^i denote the matrix with entries given by coefficients of $\big|\mu_j^i\big\rangle$ in the bases $|f_k\rangle \otimes |h_l\rangle$. We further introduce $\big\{\xi_j^i\big\}$, $\big\{\eta_j^i\big\}$,

$$\xi_j^i = \mathrm{Tr}_3\big|\mu_j^i\big\rangle\big\langle\mu_j^i\big| = B_j^i {B_j^i}^\dagger, \qquad \eta_j^i = \big(\mathrm{Tr}_2\big|\mu_j^i\big\rangle\big\langle\mu_j^i\big|\big)^* = {B_j^i}^\dagger B_j^i, \qquad j = 1, \ldots, m_i.$$

Let Γ denote a class of tripartite mixed states satisfying

$$\left[\rho_i, \rho_k\right] = 0, \qquad \left[\theta_i^{23}, \theta_k^{23}\right] = 0 \tag{4}$$

with θ_i^{23} being full rank matrices, $i, k = 1, 2, \ldots, n$, and

$$\left[\xi_t^i, \xi_l^k\right] = 0, \qquad \left[\eta_t^i, \eta_l^k\right] = 0 \tag{5}$$

with ξ_t^i being full rank matrices, $\forall i, k = 1, 2, \ldots, n, t = 1, 2, \ldots, m_i, l = 1, 2, \ldots, m_k$.

Theorem 3. *Two mixed states in Γ are equivalent under local unitary transformations if and only if the following invariants ((g) or (h)) have the same values for both mixed states:*

(g) $\quad \mathrm{Tr}(\rho_i)^\alpha, \quad \mathrm{Tr}\left(\xi_l^k\right)^\alpha, \quad \mathrm{Tr}(\rho^\gamma), \quad \alpha = 1, 2, \ldots, M, \quad \gamma = 1, 2, \ldots, MN.$

(h) $\quad \mathrm{Tr}(\theta_i^{23})^\beta, \quad \mathrm{Tr}\left(\eta_l^k\right)^\beta, \quad \mathrm{Tr}(\rho^\gamma), \quad \beta = 1, 2, \ldots, N, \quad \gamma = 1, 2, \ldots, MN.$

Proof. If ρ and $\rho' \in \Gamma$ are equivalent under the local unitary transformation $u \otimes v \otimes w$, $\rho' = u \otimes v \otimes w \rho u^\dagger \otimes v^\dagger \otimes w^\dagger$, then $|\nu_i'\rangle = u \otimes V|\nu_i\rangle$, where $V = v \otimes w$, namely A_{1i} is mapped to $A_{1i}' = uA_{1i}V^t$. Therefore,

$$\theta'^{23}_i = A'_{1i}A'^{\dagger}_{1i} = uA_{1i}A^{\dagger}_{1i}u^\dagger = u\theta_i^{23}u^\dagger,$$

$$\rho_i' = \left(A'^{\dagger}_{1i}A'_{1i}\right)^* = V\left(A^{\dagger}_{1i}A_{1i}\right)^* V^\dagger = V\rho_i V^\dagger = v \otimes w\rho_i v^\dagger \otimes w^\dagger.$$

Thus ρ_i and ρ_i' are equivalent under the local unitary transformation $v \otimes w$; from the results of bipartite system [26] we have $\mathrm{Tr}\left(\xi_l^k\right)^\alpha = \mathrm{Tr}\left(\xi'^k_l\right)^\alpha$ and $\mathrm{Tr}\left(\eta_l^k\right)^\beta = \mathrm{Tr}\left(\eta'^k_l\right)^\beta$. Therefore (g) and (h) hold.

Conversely, $\mathrm{Tr}(\rho^\gamma) = \mathrm{Tr}(\rho'^\gamma)$ imply that ρ and ρ' have the same eigenvalues. We now prove that there exist common unitary matrices V_1, V_2, V_3 such that $|\nu_i'\rangle = V_1 \otimes V_2 \otimes V_3|\nu_i\rangle$ by using lemma 2 in [15].

From the relation $\mathrm{Tr}(\rho_i)^\alpha = \mathrm{Tr}(\rho_i')^\alpha$ in (g) and condition (4), we have common unitary matrices U_1 and U_{23} for all i such that $|\nu_i'\rangle = U_1 \otimes U_{23}|\nu_i\rangle$.

The relation $\mathrm{Tr}\left(\xi_l^k\right)^\alpha = \mathrm{Tr}\left(\xi'^k_l\right)^\alpha$ in (g) and condition (5) imply that ρ_i and ρ_i' are equivalent under local unitary transformations, $\rho_i' = U_i \otimes V_i\rho_i U_i^\dagger \otimes V_i^\dagger$, according to the results of bipartite system [26]. For the case $i \neq k$ in condition (5), (5) implies that there exist common unitary matrices U and V such that $\rho_i' = U \otimes V\rho_i U^\dagger \otimes V^\dagger$. To elucidate this we just show the case $n = 2$. For a rank-two state ρ we have

$$\rho_1 = \sum_{j=1}^{m_1}\alpha_j^1|\mu_j^1\rangle\langle\mu_j^1|, \qquad \rho_2 = \sum_{j=1}^{m_2}\alpha_j^2|\mu_j^2\rangle\langle\mu_j^2|.$$

$\mathrm{Tr}\left(\xi_j^1\right)^\alpha = \mathrm{Tr}\left(\xi'^1_j\right)^\alpha$ implies that ξ_j^1 and ξ'^1_j are equivalent under unitary transformations. Therefore B_j^1 and B'^1_j have the same singular values. Equations

$$\left[\xi_t^1, \xi_l^1\right] = 0 \tag{6}$$

and

$$\left[\eta_t^1, \eta_l^1\right] = 0 \tag{7}$$

imply that (from singular value decomposition) there exist common unitary matrices U_1, U_1' and V_1, V_1' such that

$$U_1B_j^1V_1 = U_1'B'^1_jV_1'. \tag{8}$$

While

$$\left[\xi_t^2, \xi_l^2\right] = 0 \tag{9}$$

and

$$\left[\eta_t^2, \eta_l^2\right] = 0 \tag{10}$$

imply that there exist common unitary matrices U_2, U_2' and V_2, V_2' such that

$$U_2 B_j^2 V_2 = U_2' B'^2_j V_2'. \tag{11}$$

From (6) and (9), we have $U_1 = U_2$. From (7) and (10), we have $V_1 = V_2$. Hence $B'^i_j = U B_j^i V^t$ and $\left|\mu'^i_j\right\rangle = U \otimes V \left|\mu_j^i\right\rangle$, $j = 1, \ldots, m_i$. Therefore, $\rho_i' = U \otimes V \rho_i U^\dagger \otimes V^\dagger$. Hence ρ_i' and ρ_i are equivalent under local unitary transformations.

Therefore, from lemma 2 in [15] we have that tripartite states $|\nu_i\rangle$ and $|\nu_i'\rangle$ are equivalent under local unitary transformations. In fact, there exist common unitary matrices $V_i, i = 1, 2, 3$, such that $|\nu_i'\rangle = V_1 \otimes V_2 \otimes V_3 |\nu_i\rangle$, where $V_1 = W U_1, V_2 = U, V_3 = V$ $\left(\left[\theta_i^{23}, \theta_j^{23}\right] = 0\right)$ imply that there exists common W for different ν_i). Therefore, we have $\rho' = V_1 \otimes V_2 \otimes V_3 \rho V_1^\dagger \otimes V_2^\dagger \otimes V_3^\dagger$.

Thus from (g) we get that ρ and ρ' are equivalent under local unitary transformations. One can similarly prove that ρ and ρ' are equivalent under local unitary transformations from (h). □

We have discussed the local invariants for arbitrary dimensional tripartite quantum mixed states in $\mathbb{C}^K \otimes \mathbb{C}^M \otimes \mathbb{C}^N$ composite systems and have presented sets of invariants under local unitary transformations for some classes of tripartite mixed states. The invariants in a set are not necessarily independent, but they are sufficient to judge if two states in Γ or Γ_i, $i = 1, 2, 3$, are equivalent under local unitary transformations. For three qubit case, $K = M = N = 2$, a set of invariants has been presented in [10, 11] for a special class of states. By using the method in [14, 15], the results can be generalized to detect local equivalence for some special classes of general multipartite states.

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